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Third generation of vortex identification methods: Omega and Liutex/Rortex based systems *

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Abstract: A vortex is intuitively recognized as the rotational/swirling motion of fluids, but a rigorous and universally-accepted definition is still not available. Vorticity tube/filament has been regarded equivalent to a vortex since Helmholtz proposed the concepts of vorticity tube/filament in 1858 and the vorticity-based methods can be categorized as the first generation of vortex identification methods. During the last three decades, a lot of vortex identification methods, including Q , Δ, λ_2 and λ_{ci} criteria, have been proposed to overcome the problems associated with the vorticity-based methods. Most of these criteria are based on the Cauchy-Stokes decomposition and/or eigenvalues of the velocity gradient tensor and can be considered as the second generation of vortex identification methods. Starting from 2014, the Vortex and Turbulence Research Team at the University of Texas at Arlington (the UTA team) focus on the development of a new generation of vortex identification methods. The first fruit of this effort, a new omega (Ω) vortex identification method, which defined a vortex as a connected region where the vorticity overtakes the deformation, was published in 2016. In 2017 and 2018, a Liutex (previously called Rortex) vector was proposed to provide a mathematical definition of the local rigid rotation part of the fluid motion, including both the local rotational axis and the rotational strength. Liutex/Rortex is a new physical quantity with scalar, vector and tensor forms exactly representing the local rigid rotation of fluids. Meanwhile, a decomposition of the vorticity to a rotational part namely Liutex/Rortex and an anti-symmetric shear part (RS decomposition) was introduced in 2018, and a universal decomposition of the velocity gradient tensor to a rotation part ($\boldsymbol{R}$) and a non-rotation part ($\boldsymbol{NR}$) was also given in 2018 as a counterpart of the traditional Cauchy-Stokes decomposition. Later in early 2019, a Liutex/Rortex based Omega method called Omega-Liutex, which combines the respective advantages of both Liutex/Rortex and Omega methods, was developed. And a latest objective Omega method, which is still under development, is also briefly introduced. These advances are classified as the third generation of vortex identification methods in the current paper. To elaborate the advantages of the third-generation methods, six core issues for vortex definition and identification have been raised, including: (1) the absolute strength, (2) the relative strength, (3) the rotational axis, (4) the vortex core center location, (5) the vortex core size, (6) the vortex boundary. The new third generation of vortex identification methods can provide reasonable answers to these questions, while other vortex identification methods fail to answer all questions except for the approximation of vortex boundaries. The purpose of the current paper is to summarize the main ideas and methods of the third generation of vortex identification methods rather than to conduct a comprehensive review on the historical development of vortex identification methods.

Key words: Vortex definition, vortex identification, Omega, Liutex/Rortex, direct numerical simulation, turbulence

Chaoqun Liu' Short Vita

Dr. Chaoqun Liu received both Bachelor of Science (BS) degree in Fluid Mechanics (1968) and Master of Science (MS) degree in Computational Fluid Dynamics (1981) from Tsinghua University, Beijing, China and Ph. D. degree in Applied Mathematics (1989) from University of Colorado at Denver, USA. He is currently the Tenured and Distinguished Professor, a highly honored academic title, and the Director of Center for Numerical Simulation and Modeling at University of Texas at Arlington, Arlington, Texas, USA. He has worked on high order direct numerical simulation (DNS) and large eddy simulation (LES) for flow transition and turbulence for almost 30 years since 1989. As PI, he has been awarded by NASA, US Air Force and US Navy with 50 federal research grants of over 5.7×10^6 US dollars since 1990 in the United States. As Co-PI, he received 7.3×10^6 US Dollars from AFOSR in 2017. He has published 11 professional books, 107 journal papers and 145 conference papers. He chaired the first and third US AFOSR International Conference on DNS/LES in 1997 and 2001. As a principal lecturer, he held TGS2015 Workshop, New Theory of Turbulence Generation and Sustenance, Tsinghua University, Beijing, China, June 4-6, 2015, sponsored by 20 research institutions with over 220 audiences. He also organized many other conferences or workshops as the chairman or principal lecturer. He is a well-established expert in high order DNS/LES for flow transition, turbulence, and shock and boundary layer interaction. Since 2014, he has focused on the research of the vortex definition and identification. He is the founder and major contributor of the third generation of vortex identification methods including the Omega method, Liutex/Rortex method, Omega-Liutex method, RS vorticity decomposition and R-NR velocity gradient decomposition.

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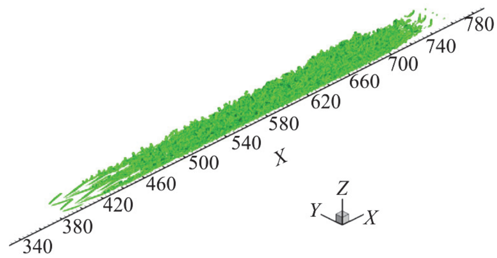
Introduction

It is widely recognized that vortices are ubiquitous in nature, from tornado, hurricane, turbulence to galaxy (Fig. 1). Vortices are considered as the building blocks of turbulent flows since turbulent flows contain countless vortices with variety of sizes and strengths. Therefore, an unambiguous and rigorous definition of vortex is indispensable for turbulence research. Intuitively, a vortex is recognized as the rotational/swirling motion of fluids. However, a universally-accepted definition of a vortex is yet to be achieved, which is probably one of the major obstacles causing considerable confusions and misunderstandings in turbulence research[1-2]. In the vast majority of research papers and textbooks, vorticity tube/filament is regarded equivalent to a vortex and the magnitude of the vorticity is considered as the measure of the local rotational strength since Helmholtz proposed the concepts of vorticity tube/ filament in 1858, which has given rise to considerable misunderstandings and confusions about vortices, especially vortex structures in turbulence. In viscous flows, vortices cannot generally be represented by vorticity tubes. Robinson[3] found that the association between regions of strong vorticity and actual vortices can be rather weak. Since then, many vortex identification criteria have been proposed[4-8]. While a certain success has been achieved, these traditional vortex identification methods have at least several shortcomings: (1) the physical meaning is not very clear, (2) a threshold is required but no one knows whether the specified threshold is proper or improper, (3) these methods are uniquely determined by the eigenvalues of the velocity gradient tensor, which could be seriously contaminated by shear, (4) they cannot capture both strong and weak vortices simultaneously. These issues prompt Liu et al. to propose a new Omega (Ω) vortex identification method[9-10]. The Omega method has several advantages: (1) simple implementation, (2) clear physical meaning (the vorticity overtakes the deformation), (3) non-dimensional and normalized, (4) robust to moderate threshold change, (5) capable to capture both strong and weak vortices simultaneously. Zhang et al. focus on the industrial applications of the Ω method, such as the application in reversible pump turbines[11-12]. In Ref. [9], it is clearly pointed out that vortices cannot be represented by the vorticity and the vorticity must be decomposed to a rotational part and a non-rotational part. However, the explicit expressions of the rotational part and the non-rotational part were not provided in Ref. [9]. Recently, a new vector called "Rortex" is introduced by Liu et al. to represent the local rigid rotation of fluids[13-14]. The name is later changed to "Liutex". Liutex/Rortex is a mathematical definition of the rigid rotational part without shear contamination. Based on the definition of Liutex/Rortex, the missing relation between the vorticity and Liutex/Rortex, namely $\nabla \times \boldsymbol{V} = \boldsymbol{R} + \boldsymbol{S}$ (the vorticity consists of Liutex/Rortex and the antisymmetric shear), can be obtained as a matter of course. In addition, the velocity gradient tensor has been decomposed to a rotational part ($\boldsymbol{R}$) and a non-rotational part ($\boldsymbol{NR}$) [14]. The new velocity gradient decomposition provides more clear physical meanings in comparison with the traditional Cauchy-Stokes decomposition which decomposes the velocity gradient tensor to a symmetric part and an antisymmetric part. Since the antisymmetric part cannot represent the flow rotation or the vorticity cannot represent a vortex, the Cauchy-Stokes decomposition is not appropriate to represent the fluid motion regarding rotation. Liutex/Rortex has a systematical definition including a scalar form for the rotation strength, a vector from for the rotation axis and a tensor form for the fluid motion decomposition. A new Omega-Liutex method, by using the Omega idea to normalize Liutex/Rortex, has also been developed[15]. And a latest objective Omega method, which can be applied in any rotating reference, is under development[16]. These advances are classified as the third generation of vortex identification methods in the current paper. The core issues for vortex identification concern with (1) the absolute strength, (2) the relative strength, (3) the rotation axis, (4) the vortex core center location, (5) the vortex core size, (6) the vortex boundary, especially in the case of the mixture of strong and weak vortices. Only the new third generation of vortex identification methods can reasonably answer these questions, while other vortex identification methods fail to answer all questions except for the approximation of vortex boundaries.

Although significant progresses have been achieved by using the third generation of vortex identification methods, there is still a long way waiting researchers to go. Since a vortex in nature is always a mixture of rotation and deformation, but Liutex/Rortex represents the rigid rotation part of the fluid motion, the precise relationship between the actual vortex and Liutex/Rortex is still unclear. In addition to the problem with the empirical parameter ε in the Omega method, what is the exact boundary of a vortex, what is the size of a vortex, what is the strength of a vortex and even what is the size of the vortex core, etc. are all open questions. Furthermore, Liutex/Rortex is expected to be applied to turbulence research, including the mechanism of turbulence generation and sustenance, Liutex/Rortex structures in turbulence, the correlation between fluctuation and Liutex/Rortex, the correlation between turbulence energy spectrum and Liutex/Rortex, large vortex formation, multiple level vortex formation, the correlation between Reynolds stress and $\boldsymbol{R}$ and $\boldsymbol{NR}$, etc. are all open questions for the future research.

(a) Tornado

(b) Hurricane



(c) Flow transition

(d) Galaxy

Fig. 1 Examples of vortices in nature

The current paper has no intention to conduct a comprehensive review of the historical development of vortex identification methods but focus on the analysis of some traditional vortex identification methods and the third-generation methods. More effort is devoted to the third generation of vortex identification methods, namely, Omega, Liutex/Rortex and Omega-Liutex/Rortex. For more detailed comparison of the existing vortex identification methods, one can refer to recently published review papers[12, 17-18].

The paper is organized as follows. Section 1 is an introduction of the vorticity-based first generation of vortex identification methods. Section 2 describes the second generation of vortex identification methods including the currently popular vortex identification methods and their shortcomings. In Section 3, the third generation of vortex identification methods including Omega, Liutex/Rortex, Omega-Liutex/Rortex are introduced and their advantages are elaborated. Section 4 presents a short introduction about the velocity gradient tensor decompositions. A short introduction on the objective Omega method which is still under development is given in Section 5. The mathematical analysis of the relation between the second and third generations of vortex identification methods is introduced in Section 6. Section 7 discusses the six core issues in vortex science and their possible solutions by the third generation of vortex identification methods. The conclusions are summarized at the end of the paper.

1. First generation: vorticity-based vortex identification methods

In early days since Helmholtz proposed the concepts of vorticity tube/filament in 1858[19], people in general believe that vortices consist of small vorticity tubes and the vortex strength is represented by the magnitude of the vorticity which is a mathematical definition of velocity curl, i.e., $\nabla \times \boldsymbol{v}$. The original Helmholtz's definitions involving vortex are given by his paper: "by vortex lines I denote lines drawn through the fluid mass so that their direction at every point coincides with the direction of the momentary axis of rotation of the water particles lying on it…," "by vortex filaments I denote portions of the fluid mass cut out from it by way of constructing corresponding vortex lines through all points of the circumference of an infinitely small surface element". These definitions directly lead to three vortex theorems: (1) **Helmholtz's first theorem:** the strength of a vortex filament is constant along its length, (2) **Helmholtz's second theorem:** a vortex filament cannot end in a fluid, it must extend to the boundaries of the fluid or form a closed path, (3) **Helmholtz's third theorem:** in the absence of rotational external forces, a fluid that is initially irrotational remains irrotational[20]. These theorems provide a guide to use the vorticity to identify or detect vortices for long time. Lamb[21] points out that: "If through every point of a small closed curve we draw the corresponding vortex-line, we mark out a tube, which we call a "vortex-tube". The fluid contained within such a tube constitutes what is called a "vortex-filament", or simply a "vortex"". Saffman[22] defines a vortex as a "finite volume of vorticity immersed in irrotational fluid". The use of the vorticity as the vortex strength is more clearly described by Nitsche[23] that "A vortex is commonly associated with the rotational motion of fluid around a common centerline. It is defined by the vorticity in the fluid, which measures the rate of local fluid rotation". Wu et al.[24] describe a vortex as "a connected fluid region with high concentration of vorticity compared with its surrounding". Although the use of the vorticity to detect vortex is widely adopted, an immediate counter-example is that in the laminar boundary layer, the magnitude of vorticity (due to shear) is very large, but no rotational motion (vortex) is found, leading to the conclusion: a vortex cannot be described by the vorticity. Actually, the vorticity is unable to distinguish between a real rotation region and a shear layer region. It is not uncommon that the average shear force generated by the no-slip wall is so strong in the boundary layer of a laminar flow plane that an extremely large amount of vorticity exists, but no rotation motions will be observed in some near-wall regions. And the maximum magnitude of the vorticity does not necessarily occur in the central region of vortex structures. As pointed out by Robinson[3], "the association between regions of strong vorticity and actual vortices can be rather weak in the turbulent boundary layer, especially in the near wall region". Wang et al.[25] obtain a similar result that the magnitude of the vorticity can be substantially reduced in the vortex core region near the solid wall in a flat plate boundary layer (Fig. 2). For a transitional flow over a flat plate, Figs. 3, 4 clearly show vorticity lines are not aligned along vortex structures and a vortex can appear in the region where the magnitude of the vorticity is relatively smaller than the surrounding region. All of these

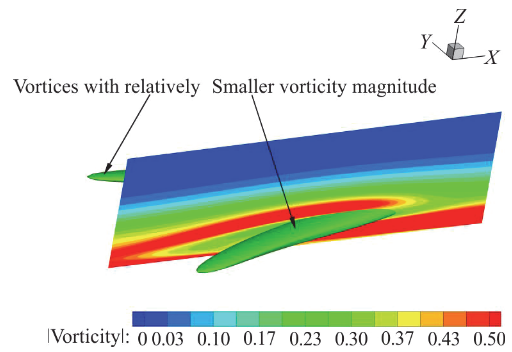
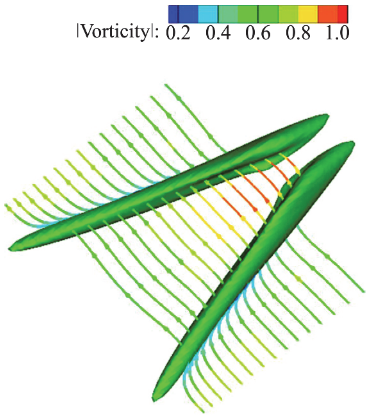
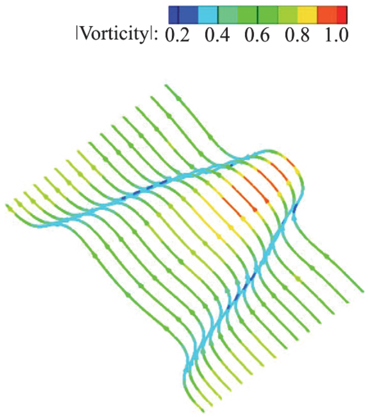


Fig. 2 Vortices in the region where the magnitude of the vorticity is relatively smaller

results indicate that the vorticity cannot be used to identify vortices.



(a) Vorticity lines with Ω iso-surfaces visible

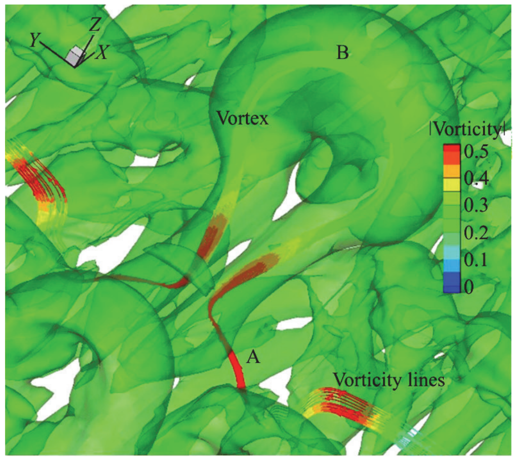


(b) Vorticity lines with Ω iso-surfaces hidden

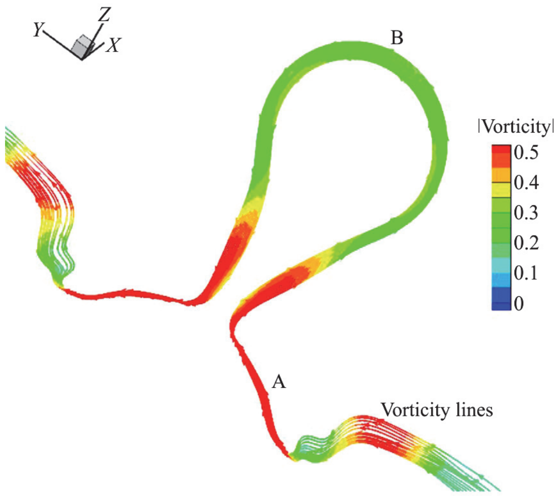
Fig. 3 Vorticity lines and Ω iso-surfaces in the region of vortex legs

2. Second generation: eigenvalue-based vortex identification methods

Another obvious candidate for vortex identification would be the one based on closed or spiraling streamlines. For example, Robinson et al.[26] point out: "A vortex exist when instantaneous streamlines mapped onto a plane normal to the vortex core exhibit a roughly circular or spiral pattern, when view from a reference frame moving with the center of the vortex core". Though it is intuitive, the determination of the vortex core is a chicken and egg problem. A more devastating drawback is that streamlines are not Galilean invariant. Lugt[27] defines: "A vortex is the rotation motion of a multitude of material particles around a common center". Lugt's definition is also intuitive, but unfortunately, still based on the pathlines of material particles which are not Galilean invariant as well. A suitable reference frame cannot be found for general unsteady 3-D flows and how to determine the common center is also vague if the pathlines are used to describe vortex structures.



(a) Vorticity lines with Ω iso-surfaces visible



(b) Vorticity lines with Ω iso-surfaces hidden

Fig. 4 Vorticity lines and Ω iso-surfaces in the region of vortex rings. The magnitude of the vorticity in the region of vortex ring (point B) is relatively smaller than that in the surrounding region (point A)

The problems associated with streamlines and/or pathlines for vortex identification motivate the rapid development of Eulerian velocity-gradient-based criteria. Most of the currently popular Eulerian vortex identification criteria are based on the analysis of the velocity gradient tensor $\nabla \boldsymbol{v}$. More specifically, these methods are exclusively dependent on the eigenvalues of the velocity gradient tensor or the related invariants. Assume that λ_1, λ_2 and λ_3 are three eigenvalues. The characteristic equation can be written as

$$\lambda^3 + P\lambda^2 + Q\lambda + R = 0 \tag{1}$$

where

$$P = -(\lambda_1 + \lambda_2 + \lambda_3) = -\mathrm{tr}(\nabla \boldsymbol{v}) \tag{2}$$

$$Q = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 = -\frac{1}{2}[\mathrm{tr}(\nabla \boldsymbol{v}^2) - \mathrm{tr}(\nabla \boldsymbol{v})^2] \quad (3)$$

$$R = -\lambda_1\lambda_2\lambda_3 = -\det(\nabla \boldsymbol{v}) \quad (4)$$

P, Q and R are three invariants of the velocity gradient tensor (Note that the symbol R will also be used to represent the magnitude of Liutex/Rortex and the meaning of the symbol R can be easily inferred from context). For incompressible flows, according to the continuity equation, $P = 0$. In the following, four representatives of the eigenvalue-based criteria, namely the Q criterion, the Δ criterion, the λ_{ci} criterion and the λ_2 criterion are briefly introduced.

2.1 Q *criterion*

The Q criterion is one of the most popular vortex identification method proposed by Hunt et al.[4]. Q is defined as the residual of the vorticity tensor norm squared subtracted from the strain-rate tensor norm squared, which can be expressed as

$$Q = \frac{1}{2}(\|\boldsymbol{B}\|_F^2 - \|\boldsymbol{A}\|_F^2) \quad (5)$$

where $\boldsymbol{A}$, $\boldsymbol{B}$ are the symmetric and antisymmetric parts of the velocity gradient tensor, respectively:

$$\boldsymbol{A} = \frac{1}{2}(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^{\mathrm{T}}) = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & \frac{\partial w}{\partial z} \end{bmatrix} \quad (6)$$

$$\boldsymbol{B} = \frac{1}{2}(\nabla \boldsymbol{v} - \nabla \boldsymbol{v}^{\mathrm{T}}) = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) & 0 & \frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) & 0 \end{bmatrix} \quad (7)$$

and $\|\cdot\|_F^2$ represents the Frobenius norm. A region with $Q > 0$ can be thought as a vortex and a second condition, which requires the pressure in the vortex region to be lower than the ambient pressure, is often omitted. However, a threshold $Q_{\mathrm{threshold}}$ must be used to define the region with $Q > Q_{\mathrm{threshold}}$ as a vortex in practice.

2.2 Δ *criterion*

Using critical point theory, Chong et al.[5] define a vortex core to be the region where $\nabla \boldsymbol{v}$ has complex eigenvalues. In a non-rotating reference frame translating with a fluid particle, the instantaneous streamline pattern (obtained from Taylor series expansion of the local velocity to a linear order) is governed by the eigenvalues of $\nabla \boldsymbol{v}$. These streamlines are closed or spiralling if two of the eigenvalues form a complex conjugate pair (for both compressible and incompressible flows). In an unsteady flow, the usage of instantaneous streamlines implies assuming the velocity field to be frozen at that instant in time. The discriminant of the characteristic equation for the velocity gradient tensor $\nabla \boldsymbol{v}$ is $\Delta = (\tilde{Q}/3)^3 + (\tilde{R}/2)^2$, $\tilde{Q} = Q - P^2/3$, $\tilde{R} = R + 2P^3/27 - PQ/3$. If $\Delta \le 0$, the three eigenvalues of $\nabla \boldsymbol{v}$ are real, if $\Delta > 0$, there exist one real eigenvalue and two conjugate complex eigenvalues and then the point is inside a vortex. Note that for incompressible flows, $P = 0$, so the following results can be obtained: $\tilde{Q} = Q$, $\tilde{R} = R$ and $\Delta = (Q/3)^3 + (R/2)^2$.

$Q > 0$ implies $\Delta > 0$, which means that the point with $Q > 0$ is inside a vortex. However, if $Q < 0$, since $(R/2)^2 > 0$, it is possible that $\Delta > 0$. Therefore, a point with $Q < 0$ is still possible to be inside a vortex, based on the Δ criterion. This implies the inconsistence between the Q and Δ criteria.

2.3 λ_{ci} *criterion*

The λ_{ci} criterion[7-8] is an extension of the Δ criterion and identical to the Δ criterion when zero threshold is applied. When the velocity gradient tensor $\nabla \boldsymbol{v}$ has two conjugate complex eigenvalues, the local time-frozen streamlines exhibit a swirling flow pattern. In this case, the eigen decomposition of $\nabla \boldsymbol{v}$ will give

$$\nabla \boldsymbol{v} = [\boldsymbol{v}_r \quad \boldsymbol{v}_{cr} \quad \boldsymbol{v}_{ci}] \begin{bmatrix} \lambda_r & 0 & 0 \\ 0 & \lambda_{cr} & \lambda_{ci} \\ 0 & -\lambda_{ci} & \lambda_{cr} \end{bmatrix} [\boldsymbol{v}_r \quad \boldsymbol{v}_{cr} \quad \boldsymbol{v}_{ci}]^{-1} \quad (8)$$

here $(\lambda_r, \boldsymbol{v}_r)$ is the real eigenpair and $(\lambda_{cr} \pm \mathrm{i}\lambda_{ci}$, $\boldsymbol{v}_{cr} \pm \mathrm{i}\boldsymbol{v}_{ci})$ the complex conjugate eigenpairs. In the local curvilinear coordinate system (c_1, c_2, c_3) spanned by the eigenvector $(\boldsymbol{v}_r, \boldsymbol{v}_{cr}, \boldsymbol{v}_{ci})$, the instantaneous streamlines are the same as pathlines and can be written as

$$c_1(t) = c_1(0)\mathrm{e}^{\lambda_r t} \tag{9}$$

$$c_2(t) = [c_2(0)\cos(\lambda_{ci}t) + c_3(0)\sin(\lambda_{ci}t)]\mathrm{e}^{\lambda_{cr}t} \tag{10}$$

$$c_3(t) = [c_3(0)\cos(\lambda_{ci}t) - c_2(0)\sin(\lambda_{ci}t)]\mathrm{e}^{\lambda_{cr}t} \tag{11}$$

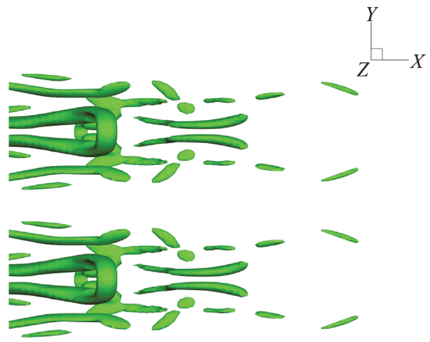
where t represents the time-like parameter and the constants $c_1(0)$, $c_2(0)$ and $c_3(0)$ are determined by the initial conditions. From Eqs. (10), (11), the period of orbit of a fluid particle is $2\pi/\lambda_{ci}$, so the imaginary part of the complex eigenvalue λ_{ci} is called the swirling strength.

2.4 λ_2 *criterion*

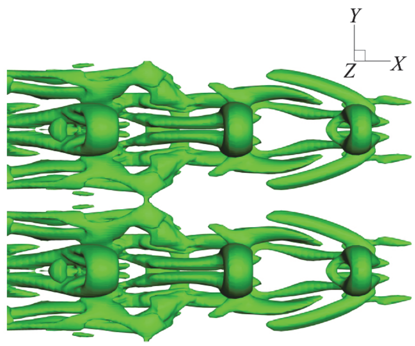
The λ_2 criterion[6] is formulated based on the observation that a local pressure minimum in a plane fails to identify vortices under strong unsteady and viscous effects. By neglecting these unsteady and viscous effects, the symmetric part of the gradient of the incompressible Navier-Stokes equation can be expressed as $\boldsymbol{A}^2 + \boldsymbol{B}^2 = -\nabla(\nabla p)/\rho$, where p is the pressure. To capture the region of local pressure minimum in a plane, Jeong and Hussain[6] define the vortex core as a connected region with two negative eigenvalues of the symmetric tensor $\boldsymbol{A}^2 + \boldsymbol{B}^2$. If the eigenvalues of the tensor $\boldsymbol{A}^2 + \boldsymbol{B}^2$ are ordered as $\lambda_1 \ge \lambda_2 \ge \lambda_3$, this definition is equivalent to the requirement that $\lambda_2 < 0$. Note that, generally, λ_2 cannot be expressed in terms of the eigenvalues of the velocity gradient tensor. However, in the special case when the eigenvectors are orthonormal, λ_2 can be exclusively determined by the eigenvalues.

2.5 *Threshold problems and problems with the rotational direction and strength*

Usually, these criteria require user-specified thresholds. It is vital to determine an appropriate threshold, since different thresholds will indicate different vortex structures. For instance, even if the same DNS data on the late boundary layer transition are examined, "vortex breakdown" will be exposed with the use of a large threshold for the Q criterion (Fig. 5(a)), while no "vortex breakdown" will be observed with a small threshold (Fig. 5(b)). The appropriate threshold is required by all of the above vortex identification methods to identify vortex structures properly. Many computational results show that the threshold is sensitive, case-related, time step-related, empirical and hard to adjust. More seriously, actually no one knows whether the specified threshold is proper or improper, but different thresholds would present different vortex structures. There is no single proper threshold especially if strong and weak vortices co-exist. If the threshold is too large, weak vortices will be wiped out. If the threshold is too small, weak vortices may be captured, but strong vortices could be smeared and become vague. The other disadvantage of these vortex identification methods is that only iso-surface can be displayed by these vortex identification methods, but no rotational axis or vortex direction can be obtained since they are only scalar criteria. Naturally, one question emerges: can the iso-surface represent the rotational strength? The answer is no, since they are different to each other (not unique), contaminated by shear in different degrees[14], and fail to represent the rigid rotation of the fluid motion.



(a) "Vortex breakdown" with a large threshold (Q=0.024)



(b) No "vortex breakdown" with a small threshold (Q=0.002)

Fig. 5 Different vortex structures identified by the Q criterion with different thresholds

3. Third generation: Omega, Liutex, Omega-Liutex

Starting from 2014, researchers at the University of Texas at Arlington have been devoted to developing new vortex identification methods to overcome the above-mentioned shortcomings associated with the traditional vortex identification methods. The first breakthrough was made in 2016 when Liu et al. published the new Omega vortex identification method[9]. The Omega method was later modified by Dong et al.[10], based on a new formula to determine ε.

3.1 *Omega vortex identification method*

The Omega method was originated from an important physical understanding that a vortex is a region where the vorticity overtakes the deformation. The vorticity cannot directly represent the fluid rotation, although there is no rigid rotation without vorticity. Therefore, the vorticity could be small in the region with strong rotation and could be large in the region with weak or zero rotation. The Blasius boundary layer is a typical example. The deformation is also an important factor in a rotational flow while a vortex presents. Therefore, it is reasonable to consider the ratio of the vorticity and the deformation for vortex identification. As given in Ref. [9], Ω is defined as a ratio of the vorticity tensor norm squared over the sum of the vorticity tensor norm squared and deformation tensor norm squared

$$\Omega = \frac{\|\boldsymbol{B}\|_F^2}{\|\boldsymbol{A}\|_F^2 + \|\boldsymbol{B}\|_F^2} = \frac{b}{a+b} \tag{12}$$

where $a = \|\boldsymbol{A}\|_F^2$, $b = \|\boldsymbol{B}\|_F^2$. In practice, a small positive parameter ε is added to the denominator of Eq. (12) to avoid non-physical noises, so Ω can be expressed as

$$\Omega = \frac{b}{a+b+\varepsilon} \tag{13}$$

By way of metaphor, Ω looks like a mixture of salt and water. The salty is measured by the ratio of salt and water, not by salt itself. From this example, Ω may be considered as the density of the vorticity. More specifically, Ω is the rigidity of the fluid motion. Apparently, the fluid will become a rigid body if $\Omega = 1$ which indicates no deformation.

3.2 *The robustness of the Omega method*

The Omega method requires a parameter larger than 0.5 as the threshold. In practice, $\Omega = 0.51$ or $\Omega = 0.52$ can be used as the fixed threshold. Actually, the Omega method is insensitive to the moderate threshold change. For the DNS case of flow transition and the LES case of shock and boundary layer interaction, vortex structures identified by the Omega method are very similar when $\Omega = 0.52 \sim 0.60$ [9]. The threshold $\Omega = 0.52$ is reliable and can present the periphery of vortices for many different cases. This conclusion has been confirmed by many Omega users. One of the major shortcomings of the traditional vortex identification methods is highly dependent on the proper selection of the threshold and very sensitive to the threshold change. In contrast, the Omega method is very insensitive to the moderate threshold change. This is one of distinct advantages of the Omega method over other vortex identification methods. Actually, the shape of vortex structures will be merely fatter or thinner with the moderate change of the threshold. However, the selection of $\Omega = 0.52$ is only empirical for the approximation of vortex boundaries and higher threshold, such as 0.8, 0.9, etc., may be needed if the vortex cores will be tracked.

3.3 *Determination of* ε

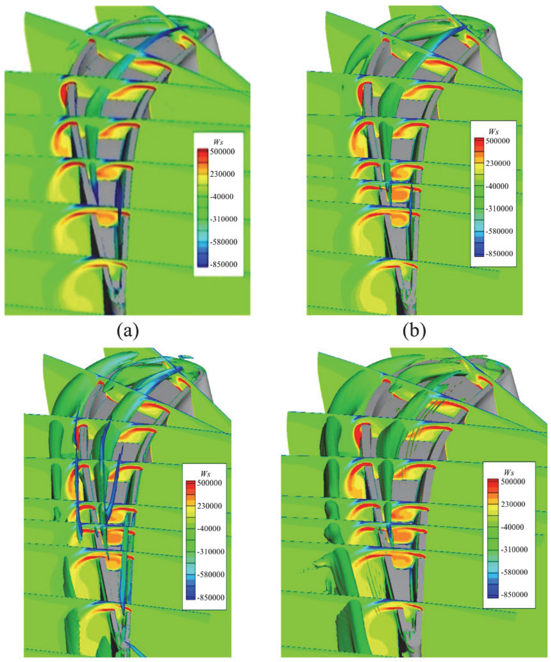
In the original definition of Ω, ε is a small positive parameter used to avoid division by zero. Although Ω is non-dimensional and normalized as $\Omega \in [0,1]$, some serious noises may appear inside the flow domain if both terms a, b are close to zero. These noises can be reduced or even removed by introducing a proper positive number ε in the denominator of Ω. However, the proper value of ε is still case-related and dimension-related. Apparently, ε is strongly related to the cases and timesteps, especially if the dimensional governing equations are applied. Even for the non-dimensional governing equations, the reference length and reference speed could be very different in different cases. A proper selection of ε is critical. An empirical approach to determine ε is proposed in Ref. [10], based on a large number of test results from different cases. ε is defined as a function of $(b-a)_{\max}$ and can be written as

$$\varepsilon = 0.001 \times (b-a)_{\max} = 0.002 \times Q_{\max} \tag{14}$$

The term $(b-a)_{\max}$ represents the maximum of the difference of the vorticity squared and the deformation squared and is easy to obtain at each timestep in a certain case. Therefore, the manual adjustment of ε in many cases can be avoided.

3.4 *Capability of capturing both strong and weak vortices simultaneously*

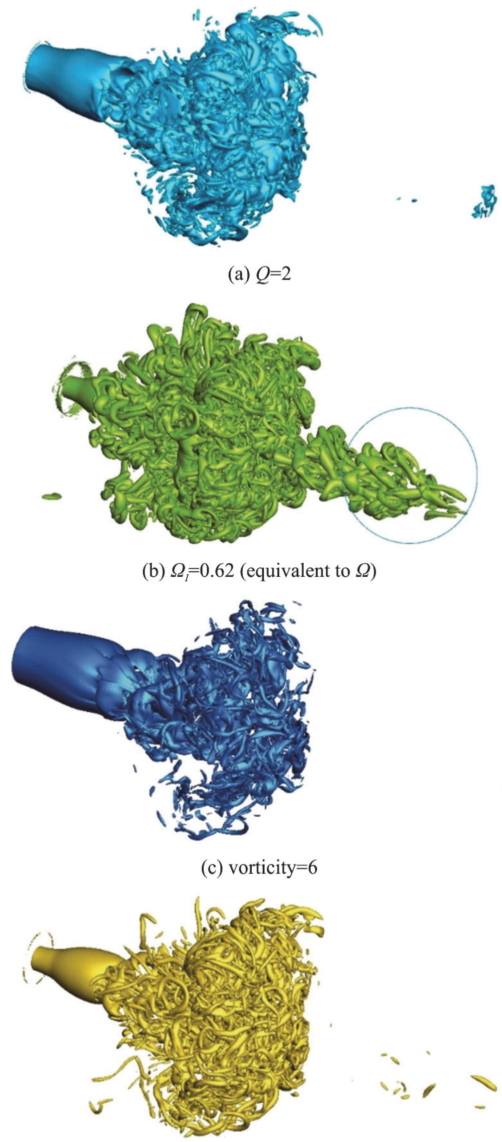
When there exist both strong and weak vortices, the Omega method can capture both strong vortices and weak vortices simultaneously, while other vortex identification methods fail. Following are two examples provided by the research groups in Beihang University and Tsinghua University respectively through courtesy personal contacts. Figure 6 shows the vortex structures around turbine tip identified by the Q, Ω criteria (provided by courtesy personal contact of Zou et al. in Beihang University). As can be seen from Figs. 6(a), 6(b), the weak vortex structures located in the left fail to be captured by the Q iso-surface with a large threshold, although the strong prime vortex in the central area is similar with the one captured by the Omega method $(\Omega = 0.52)$, however, when the threshold of Q is set to a smaller value (Figure 6(c)), the weak vortices are basically captured but the prime vortex is smeared by the peripheral structures. Therefore, it is obvious that the Omega method can successfully capture both strong and weak vortex structures simultaneously.



(a) (b) (c) (d)

Fig. 6 Comparison of Q iso-surfaces and Ω iso-surfaces of vortex structures around turbine tip: (a) Large threshold (no weak vortices) $(Q = 10^{10})$, (b) Medium threshold (blurry weak vortices) $(Q = 5\times 10^{9})$, (c) Small threshold (smeared prime vortex) $(Q = 10^{9})$ and (d) Ω (clear strong and weak vortex structures) $(\Omega = 0.52)$ (provided by courtesy personal contact of Zou et al.)

Figure 7 shows the vortex structure in swirling flow given by Gui et al. in Tsinghua University. Note that the swirling flow is characterized by the motion of fluid swirl imparted onto a directional jet flow or without directional jet flow. The direct numerical simulation data of swirling jets flows in a rectangular container were utilized here, and the comparison of several vortex identification methods including Q, Ω_L (equivalent to Ω), vorticity and λ_2 have been carried out to assess the performance and capacity of these methods. The conclusion is that Ω_L can identify more additional secondary weak vortices (strongly kinked weak vortices), which cannot be clearly visualized by Q, λ_2.



(a) Q=2

(b) Ω_L=0.62 (equivalent to Ω)

(c) vorticity=6

(d) λ_2=-2

Fig. 7 Vortex structures of swirling jets flows in a rectangular container

3.5 *Liutex/Rortex-an eigenvector-based vortex definition*

Although the vorticity decomposition to a rotational part and a non-rotational part has been given by Liu et al.[9], the idea of extracting the rigid rotation part from the fluid motion was first presented in the "First Workshop on Vortex and Turbulence" held in

Shanghai, China on December 15-17, 2017. The rotational part of the vorticity is defined as Liutex/Rortex vector, which can clearly represent both the direction and magnitude of the rotational motion. This vector was originally named as "Rortex" but renamed as "Liutex" in December 2018. The exact form and mathematical definition of Liutex/Rortex can be found in Liu et al.[13], Gao and Liu[14] and summarized as follows:

Definition 1 A local rotational axis is defined as the direction of $\boldsymbol{r}$ where

$$\mathrm{d}\boldsymbol{v} = \alpha \mathrm{d}\boldsymbol{r} \tag{15}$$

This definition means that there is no cross-velocity increment perpendicular to the direction of the local rotational axis. For example, if the z-axis is the rotational axis in a reference frame, the velocity can only increase or decrease along the z-axis, which means only $\mathrm{d}w \neq 0$, but $\mathrm{d}u = 0$ and $\mathrm{d}v = 0$. Accordingly, the following theorem is obtained:

Theorem 1: The direction of the local rotational axis is the real eigenvector of the velocity gradient tensor $\nabla \boldsymbol{v}$.

Proof: If $\boldsymbol{r} = [r_x, r_y, r_z]^{\mathrm{T}}$ represents the direction of the local rotational axis, we have $\mathrm{d}\boldsymbol{v} = \alpha \mathrm{d}\boldsymbol{r}$. From the definition of the velocity gradient tensor, the velocity increment can be expressed as

$$\mathrm{d}\boldsymbol{v} = \nabla \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{r} \tag{16}$$

Therefore

$$\mathrm{d}\boldsymbol{v} = \nabla \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{r} = \alpha \mathrm{d}\boldsymbol{r} \tag{17}$$

Using $\boldsymbol{r}$ to substitute $\mathrm{d}\boldsymbol{r}$, we have

$$\nabla \boldsymbol{v} \cdot \boldsymbol{r} = \lambda_r \boldsymbol{r} \tag{18}$$

which means $\boldsymbol{r}$ is the real eigenvector of $\nabla \boldsymbol{v}$, and λ_r is the real eigenvalue.

After the direction of the rigid rotational axis is obtained, the next step is to find the exact angular speed of the rigid rotation of the fluid motion as the magnitude of Liutex/Rortex. First, a coordinate rotation (Q rotation) is required to rotate the original z-axis to the direction of the local rotational axis $\boldsymbol{r}$ and the velocity gradient tensor $\nabla \boldsymbol{V}$ in the new XYZ frame will become

$$\nabla \boldsymbol{V} = \boldsymbol{Q} \nabla \boldsymbol{v} \boldsymbol{Q}^{\mathrm{T}} = \begin{bmatrix} \frac{\partial U}{\partial X} & \frac{\partial U}{\partial Y} & 0 \\ \frac{\partial V}{\partial X} & \frac{\partial V}{\partial Y} & 0 \\ \frac{\partial W}{\partial X} & \frac{\partial W}{\partial Y} & \frac{\partial W}{\partial Z} \end{bmatrix} \tag{19}$$

where $\boldsymbol{Q}$ is a rotation matrix while U, V and W represent the velocity components in the XYZ coordinate. It should be noted that there is a typing mistake in Eq. (30) of Ref. [14]. The unit vector for Q rotation is omitted by mistake. This unit vector can be written as

$$\boldsymbol{\gamma} = \begin{bmatrix} \gamma_x \\ \gamma_y \\ \gamma_z \end{bmatrix} = \frac{\boldsymbol{z} \times \boldsymbol{r}}{|\boldsymbol{z} \times \boldsymbol{r}|}, \quad \boldsymbol{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{20}$$

And the components used in Eq. (30) of Ref. [14] should be γ_x, γ_y, γ_z rather than the eigenvector components γ_x, γ_y, γ_z. Accordingly, the rotation matrix $\boldsymbol{Q}^*$ should be written as

$$\boldsymbol{Q}^* =$$

$$\begin{bmatrix} \cos\phi + \gamma_x^2 d & \gamma_x \gamma_y d - \gamma_z \sin\phi & \gamma_x \gamma_z d + \gamma_y \sin\phi \\ \gamma_y \gamma_x d + \gamma_z \sin\phi & \cos\phi + \gamma_y^2 d & \gamma_y \gamma_z d - \gamma_x \sin\phi \\ \gamma_z \gamma_x d - \gamma_y \sin\phi & \gamma_z \gamma_y d + \gamma_x \sin\phi & \cos\phi + \gamma_z^2 d \end{bmatrix} \tag{21}$$

where $d = 1 - \cos\phi$, $\phi = \arccos(\boldsymbol{z} \cdot \boldsymbol{r})$.

And then, a second rotation (P rotation) is applied to rotate the reference frame around the Z-axis and the corresponding velocity gradient tensor $\nabla \boldsymbol{V}_\theta$ can be written as

$$\nabla \boldsymbol{V}_\theta = \boldsymbol{P} \nabla \boldsymbol{V} \boldsymbol{P}^{-1} \tag{22}$$

where $\boldsymbol{P}$ is a rotation matrix around the Z-axis and can be written as

$$\boldsymbol{P} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{23}$$

Therefore, the components of 2×2 upper left submatrix of $\nabla \boldsymbol{V}_\theta$ in Eq. (22) are

$$\left.\frac{\partial U}{\partial Y}\right|_\theta = \alpha \sin(2\theta + \varphi) - \beta \tag{24}$$

$$\left.\frac{\partial V}{\partial X}\right|_\theta = \alpha \sin(2\theta + \varphi) + \beta \tag{25}$$

$$\left.\frac{\partial U}{\partial X}\right|_\theta = -\alpha \cos(2\theta + \varphi) + \frac{1}{2}\left(\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y}\right) \tag{26}$$

$$\left.\frac{\partial V}{\partial Y}\right|_\theta = \alpha\cos(2\theta+\varphi) + \frac{1}{2}\left(\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y}\right) \tag{27}$$

where

$$\alpha = \frac{1}{2}\sqrt{\left(\frac{\partial V}{\partial Y} - \frac{\partial U}{\partial X}\right)^2 + \left(\frac{\partial V}{\partial X} + \frac{\partial U}{\partial Y}\right)^2} \tag{28}$$

$$\beta = \frac{1}{2}\left(\frac{\partial V}{\partial X} - \frac{\partial U}{\partial Y}\right) \tag{29}$$

In order to get the minimum absolute value of $\partial U / \partial Y|_\theta$ as the angular speed of the rigid rotation (assuming $\beta > 0$), we must have

$$\sin(2\theta + \varphi) = 1 \tag{30}$$

The rotational strength is defined as twice the minimal absolute value of the off-diagonal component of the 2×2 upper left submatrix and is given by

$$R = 2(\beta - \alpha)\,,\quad \beta^2 > \alpha^2 \tag{31a}$$

$$R = 0\,,\quad \alpha^2 \geq \beta^2 \tag{31b}$$

Then the vector form of Liutex/Rortex is obtained by

$$\boldsymbol{R} = R\boldsymbol{r} \tag{32}$$

For detailed implementation, one can refer to Refs. [13-14]. Wang et al.[28] recently derive an explicit formula to calculate the magnitude of $\boldsymbol{R}$ and dramatically simplify the implementation. The magnitude R can be obtained as

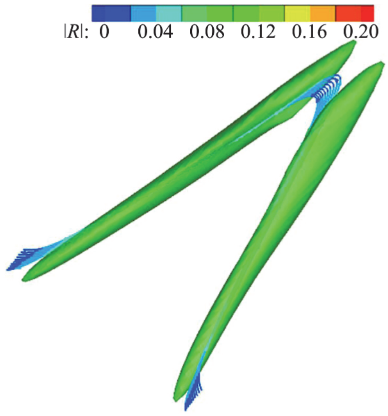
$$R = \langle \boldsymbol{\omega}, \boldsymbol{r} \rangle - \sqrt{\langle \boldsymbol{\omega}, \boldsymbol{r} \rangle^2 - 4\lambda_{ci}^2} \tag{33}$$

Thus, the Liutex vector can be defined as

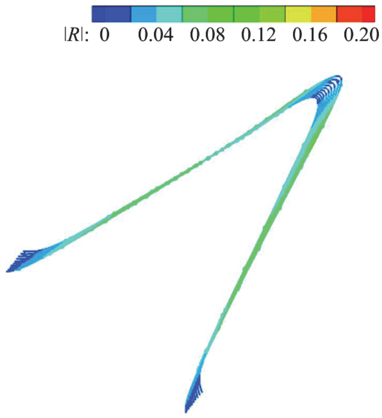
$$\boldsymbol{R} = R\boldsymbol{r} = \left\{\langle \boldsymbol{\omega}, \boldsymbol{r} \rangle - \sqrt{\langle \boldsymbol{\omega}, \boldsymbol{r} \rangle^2 - 4\lambda_{ci}^2}\right\}\boldsymbol{r} \tag{34}$$

where $\boldsymbol{\omega}$ is the vorticity vector.

Note that, being different from all other vortex identification methods introduced above, Liutex/Rortex is a vector which provides both the local rotation axis and the rigid-body angular speed. Therefore, Liutex/Rortex vectors, lines and tubes can all be applied to describe vortex structures (Figs. 8, 9) while all other vortex identification methods introduced above can only be used to show the iso-surface as a representation of the vortex boundaries, the appropriateness of which is rather questionable. In addition, the thresholds of these iso-surfaces are kind of arbitrary selection which are very different from case to case and the results are definitely contaminated by shear because the only correct quantity to represent the rotational strength is the magnitude of Liutex/Rortex.



(a) Liutex/Rortex lines with the iso-surfaces visible



(b) Liutex/Rortex lines with the iso-surfaces hidden

Fig. 8 Liutex/Rortex line representation for the Lambda vortex

3.6 *Omega-Liutex vortex identification method*

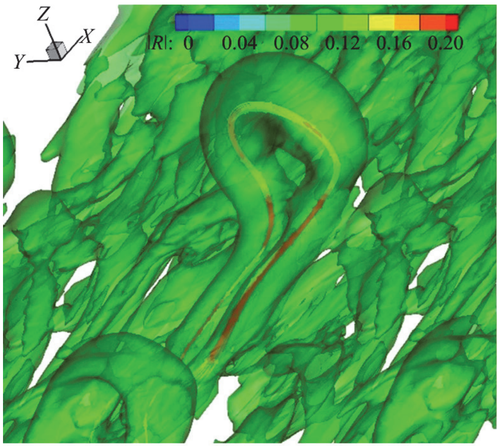
Recently, a new idea to combine the Omega and Liutex/Rortex methods called Omega-Liutex (Ω_R) is proposed[15]. Ω_R is defined as the ratio of β squared over the sum of β squared and α squared and can be written as

$$\Omega_R = \frac{\beta^2}{\alpha^2 + \beta^2 + \varepsilon} \tag{35}$$

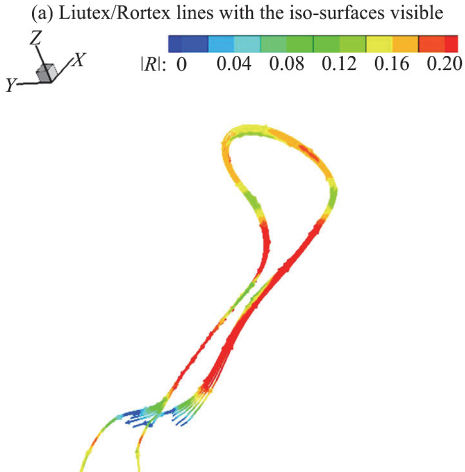
A small positive parameter ε is also introduced in the denominator of Ω_R to remove non-physical noises. Similarly, ε is empirically defined as a function of the maximum of the term $\beta^2 - \alpha^2$, proposed as follows

$$\varepsilon = b \times (\beta^2 - \alpha^2)_{\max} \tag{36}$$

where b is a small positive number, e.g., 0.001. For each case, b is a fixed parameter and the term $(\beta^2 - \alpha^2)_{\max}$ can be easily obtained at each time step and the manual adjustment of ε is avoided. Thus, the iso-surface of $\Omega_R = 0.52$ can be chosen to visualize vortex structures, which is equivalent to indicate the region where the vorticity overtakes the principal strain rate on the plane normal to the local rotational axis.



(a) Liutex/Rortex lines with the iso-surfaces visible



(b) Liutex/Rortex lines with the iso-surfaces hidden

Fig. 9 Liutex/Rortex line representation for the ring-like vortex

The new Ω_R method has the following advantages:

(1) Ω_R is able to measure the relative rotation strength on the plane perpendicular to the local rotational axis.

(2) Ω_R is a normalized function from 0 to 1 and can be further used in statistics and correlation analysis as a physical quantity.

(3) Ω_R can separate the rotational vortices from shear layers, discontinuity structures and other non-physical structures.

(4) Compared with many vortex identification methods which require case-dependent thresholds to capture the vortex structures, Ω_R is quite robust and can be always set as 0.52 to capture vortex structures in different cases and at different timesteps.

4. New velocity gradient tensor decomposition based on Liutex/Rortex

Based on Liutex/Rortex, the velocity gradient tensor decomposition can be written as

$$\nabla \boldsymbol{V} = \begin{bmatrix} \lambda_{cr} & -\phi & 0 \\ \phi + s & \lambda_{cr} & 0 \\ \xi & \eta & \lambda_r \end{bmatrix} = \boldsymbol{R} + \boldsymbol{NR} \tag{37}$$

$$\boldsymbol{R} = \begin{bmatrix} 0 & -\phi & 0 \\ \phi & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{38}$$

$$\boldsymbol{NR} = \begin{bmatrix} \lambda_{cr} & 0 & 0 \\ s & \lambda_{cr} & 0 \\ \xi & \eta & \lambda_r \end{bmatrix} \tag{39}$$

where $\boldsymbol{R}$ stands for the rotational part of the local fluid motion, which is called the tensor version of Liutex/Rortex, and $\boldsymbol{NR}$ the non-rotational part. It is clear that $\boldsymbol{NR}$ has three real eigenvalues, so $\boldsymbol{NR}$ itself implies no local rotation, here

$$\lambda_r = \frac{\partial W}{\partial Z} \tag{40}$$

$$\lambda_{cr} = \frac{1}{2}\left(\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y}\right) \tag{41}$$

$$\phi = \beta - \alpha = \frac{R}{2} \tag{42}$$

$$s = 2\alpha \tag{43}$$

$$\xi = \frac{\partial W}{\partial X} \tag{44}$$

$$\eta = \frac{\partial W}{\partial Y} \tag{45}$$

The traditional Cauchy-Stokes decomposition ignores the fundamental role of the local rotation part represented by Liutex/Rortex. It is believed that it is the nature and most important part of the fluid motion and turbulence generation. Without Liutex/Rortex, the world would not have turbulence. Therefore, the Cauchy-Stokes decomposition may not correctly represent the fluid motion decomposition because the antisymmetric tensor only represents the vorticity, but the vorticity cannot represent the fluid rotation. In such a sense, the new velocity gradient decomposition based on Liutex/Rortex is an appropriate fluid motion decomposition.

5. Preliminary introduction of the objective Omega method

Objectivity of vortex identification is also a very important topic especially in a rotating coordinate system, turbine machinery for example. Objectivity means the vortex identification method should be independent of observers. In order to retain objectivity of the vortex structure, a new objective Omega method called $\tilde{\Omega}$ method is developed by Liu et al.[16]. This method counteracts the effects of the moving framework by introducing spatially-averaged vorticity. Then net spin tensor can be defined by

$$\tilde{\boldsymbol{B}} = \boldsymbol{B} - \bar{\boldsymbol{B}} \tag{46}$$

where $\bar{\boldsymbol{B}}$ is the antisymmetric spin tensor formed by spatially-averaged vorticity. Finally, the objective Omega method can be defined by

$$\tilde{\Omega} = \frac{\|\tilde{\boldsymbol{B}}\|_F^2}{\|\boldsymbol{A}\|_F^2 + \|\tilde{\boldsymbol{B}}\|_F^2 + \varepsilon} \tag{47}$$

The objective method is still under development. As a sample example, the vortex structures after micro-vortex generator in a variable-time moving frame is demonstrated by the new developed objective Omega method (Figs. 10, 11). The data from ILES is visualized by the non-objective and objective Omega method in a moving reference frame. The results show that, if the transformations of a spatial rotation and a translation are both time-related, the original Omega method is not objective and the identified vortex structures will be totally polluted (Fig. 10). However, the $\tilde{\Omega}$ method is objective (Fig. 11).

6. Mathematical analysis of the relation between the second and third generations of vortex identification methods

In this section, several questions related to the second and third generations of vortex identification methods will be clarified.

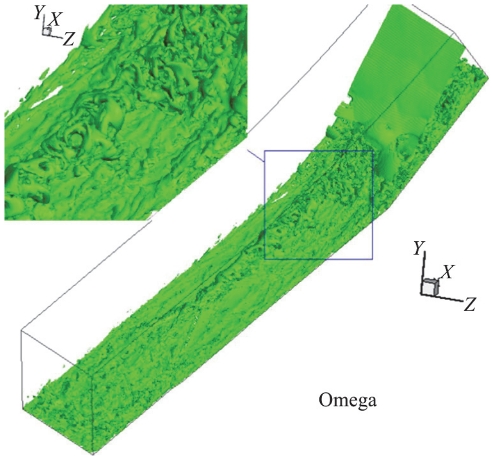


Fig. 10 Vortex structures identified by the non-objective Omega method ($\Omega = 0.52$) in a moving reference frame

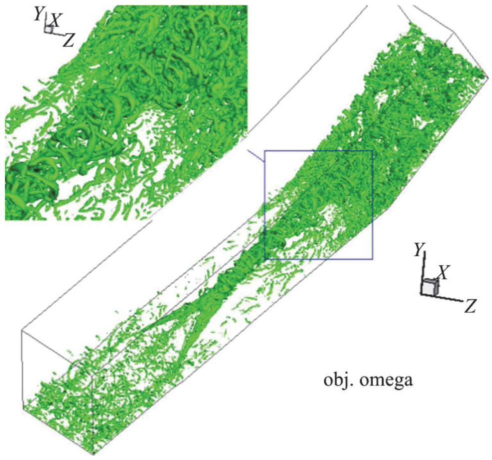


Fig. 11 Vortex structures identified by the objective Omega method ($\tilde{\Omega} = 0.52$) in a moving reference frame

6.1 *Modified Omega vortex identification method*

A modified Omega method is proposed in Ref. [25] as

$$\Omega_M = \frac{b + \varepsilon}{a + b + 2\varepsilon} \tag{48}$$

where $\varepsilon = \sigma(b - a)_{\max}$, $\sigma = 0.0005\text{-}0.0010$.

For uniform flow, $b = a = 0$, $\Omega_M = 0.5$, there exists no vortex. For weak vortices, $b \approx 0$, $a \approx 0$, but $b > a$, we will obtain $\Omega_M > 0.5$ which is consistent with the definition. Therefore, it is justified that we should add ε on the numerator and 2ε on the denominator.

However, σ is empirical and may need to be adjusted. As we repeatedly addressed, there is a question with the empirical and case-related selection of ε in the Omega method. Several computational results indicate that although the modified Omega method is better than the original version, this problem is still open for future research.

6.2 *Relation between* Q *and* Ω

In practice, the Q criterion can be written as

$$Q = \frac{1}{2}(b-a) = Q_{Th} \tag{49}$$

where Q_{Th} stands for the threshold of the Q criterion.

This is exactly equivalent to $\Omega = 0.50$ and $\varepsilon = 2Q_{Th}$ because $\Omega = b/(a+b+2Q_{Th}) = 0.50$ will lead to $b = 0.5(a+b+2Q_{Th})$ which is exactly $Q = 0.5(b-a) = Q_{Th}$.

However, if one pick $\Omega = 0.52$, $b = 0.52(a+b+\varepsilon)$ and $Q = 0.5(b-a) = 0.5(0.52b - 0.48a + 0.52\varepsilon) \neq Q_{Th}$ because the right-hand side is a variable dependent on a, b, but Q_{Th} is a fixed number. According to the discussion above, it is clear that the Q criterion is a special case of the Omega method when $\Omega = 0.50$, $\varepsilon = 2Q_{Th}$. However, if $\Omega = 0.52$ or any number greater than 0.5, the Q criterion and the Omega method are quite different. In other words, for any case, the Omega method is exactly equivalent to the Q criterion with $\Omega = 0.50$, $\varepsilon = 2Q_{Th}$, but it is impossible to find the Q criterion to be equivalent to the Omega method if $\Omega > 0.50$.

6.3 *Relation between* Q *and* Ω_M

According to Eq. (49), if $\Omega_M = 0.50$, $b + \varepsilon = 0.5(b + a + 2\varepsilon)$ and $b - a = 0$. If $\Omega_M > 0.5$, we will have $Q = 0.5(b-a) > 0$. The modified Omega method is consistent with the Q criterion. However, these two methods are never equivalent. The Q criterion is really $Q = 0.5(b-a) = Q_{Th}$ which can never be equivalent to Ω_M. For example, if $\Omega_M = 0.5 + \sigma$, we will have

$$b + \varepsilon = (0.5 + \sigma)(b + a + 2\varepsilon) \tag{50}$$

which will lead to $Q = 0.5(b-a) = \sigma(b + a + 2\varepsilon) \neq Q_{Th}$ since the right-hand side is strongly dependent on b, a, but Q_{Th} is a constant as the threshold of Q.

6.4 *Equivalence of vortex identification methods in mathematical vortex boundary*

The mathematical formula for the vortex identification can be described as follows:

$$\Delta \text{ criterion: } \Delta > 0 \tag{51}$$

$$\lambda_{ci} \text{ criterion: } \lambda_{ci} > 0 \tag{52}$$

$$\lambda_2 \text{ criterion: } \lambda_2 < 0 \tag{53}$$

$$Q \text{ criterion: } Q > 0 \tag{54}$$

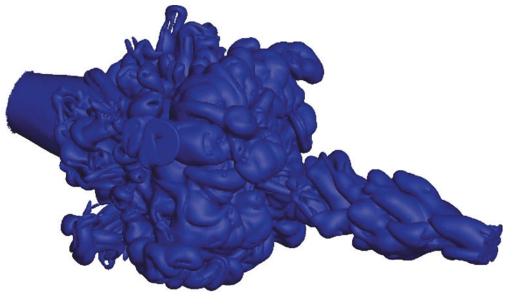
$$\Omega\text{: } \Omega > 0.5 \tag{55}$$

$$\text{Liutex/Rortex: } R > 0 \tag{56}$$

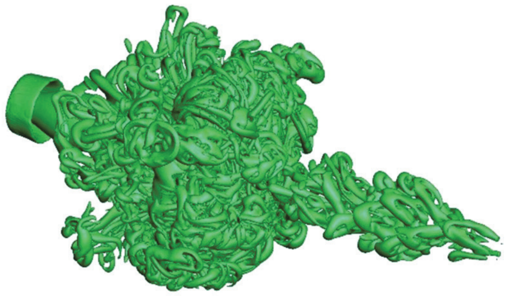
If $\Delta = 0$ at some point, we have $\lambda_{ci} = 0$ at that point. Mathematically the boundary of a vortex must have $\Delta = 0$. From this point of view, all vortex identification methods discussed above can give a similar mathematical vortex boundary. Actually, $\Delta = 0$, $\lambda_{ci} = 0$ and $R = 0$ are equivalent, and $Q = 0$, $\Omega = 0.50$ are equivalent.

6.5 *Differences to mixture of strong and weak vortices when the threshold is large*

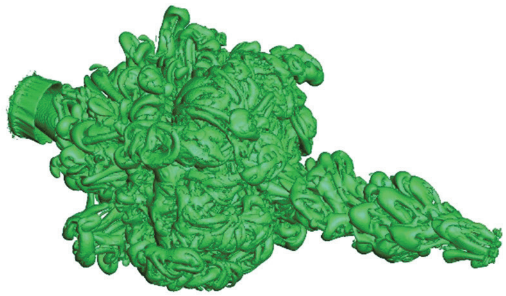
However, in practice when we use the computer, we have to pick a threshold to find the approximate vortex boundary instead of the mathematical vortex boundary, let $Q = Q_{Th}$ for example. If we pick $Q = 0$, the identified structures will be contaminated by many noises. When the threshold is small, due to the continuity of the vortex identification scalar function, the difference of the visualizations by different methods must be small except for the vorticity iso-surface. This is the reason why when the threshold is small, all the vortex identification methods present almost the same structures except for the vorticity iso-surface for a swirling jet flow (Fig. 12). However, if we want to find the vortex core centers, we have to increase the threshold, vorticity $= 6$, $Q = 3.1$ and $\Omega = 0.63$ for example. The strong vortices in the upstream have been captured by the Q and Omega methods, but the weak vortices downstream are captured only by the Omega method but disappear in the graphics identified by the vorticity and the Q criteria. Figure 13 also clearly shows the first and second generations of vortex identification methods cannot track the location and size of all of the vortex cores since the weak vortices are already removed by a large threshold. On the other hand, all vortices (no matter absolutely strong or weak) keep the existence with the shape change from fatter to thinner by using the Omega method because the Omega method only uses the relative vortex strength or, in other words, the vorticity density or fluid rigidity. From Fig. 13(d), it can be seen that the absolute strength or Liutex/Rortex is very weak in the downstream, but the Omega method can capture them very well.



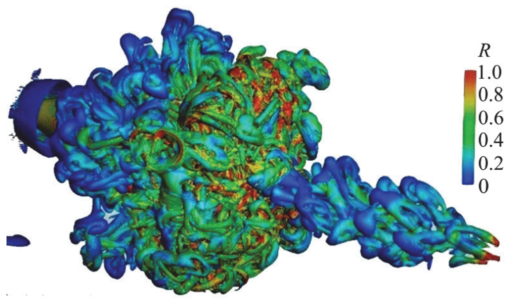
(a) First generation, Vorticity=0.5



(b) Second generation, Q=0.05

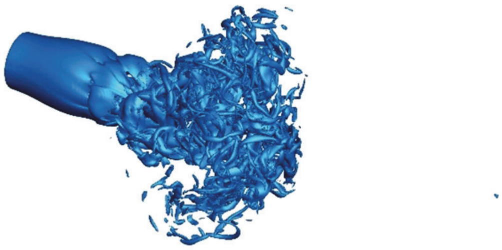


(c) Third generation, R=0.1

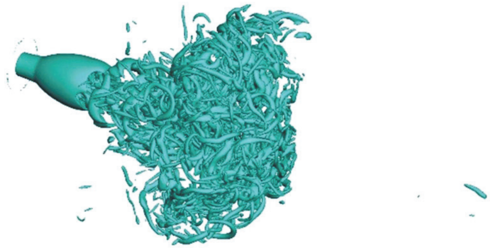


(d) Third generation, Ω=0.51

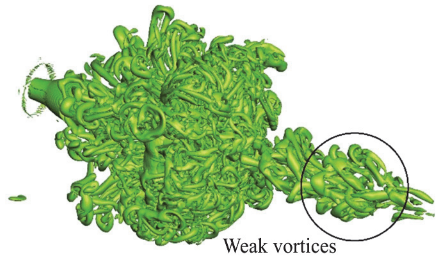
Fig. 12 Comparison of three generations of vortex identification methods with small thresholds for a swirling jet with colors showing the Liutex/Rortex magnitude



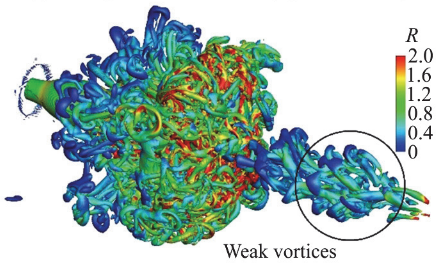
(a) First generation, Vorticity=6



(b) Second generation, Q=3.1



(c) Third generation, Ω_L=0.63 (equivalent to Ω)



(d) Third generation, Ω_L=0.63 (colors presenting liutex/Rortex magnitude)

Fig. 13 Comparison of three generations of vortex identification methods with large thresholds for a swirling jet with colors presenting the Liutex/Rotex magnitude

7. Core issues on vortex definition and identification

A vortex in fluids is always a mixture of fluid rotation and deformation and there is no rigorous mathematical definition for a vortex so far. However, there are several core issues involved in vortex identification:

(1) What is the absolute strength of a vortex?
(2) What is the relative strength of a vortex?
(3) What is the rotation axis of a vortex?
(4) Where is the vortex core center?
(5) What is the size of the vortex core?
(6) Where is the vortex boundary?

In the following, the possible answers based on the third generation of vortex identification methods are presented.

Definition 2 The absolute strength of a vortex is defined as the angular speed of the rigid rotation part of the fluid motion.

According to Definition 2, Liutex/Rortex is the only measurement of the absolute strength and other vortex identification methods are contaminated by shear. For detailed demonstration of the contamination by shear, one can refer to Ref. [14].

Definition 3 The relative strength of a vortex is defined as a ratio of the vorticity squared over the sum of the vorticity squared and the deformation squared.

According to Definition 3, Ω is the measurement of the relative vortex strength and all other second generation of vortex identification methods cannot give the relative vortex strength. The relative vortex strength may be considered as the density of the vorticity or the rigidity of fluids. Actually, the relative strength is much more important than the absolute strength which is something like that relative error is much more meaningful than the absolute error. In a flow field, a vortex may rotate at an angular speed of 1 000 circles/second and the other may rotate at an angular speed of 10 circles/second. However, a threshold of 100 circles/second will fail to identify all vortices with the angular speed less than 100 circles/second. The use of Omega, or the density of the vorticity, to vortex identification is much better than the absolute vortex strength. The Ω_R method combining Omega and Liutex/Rortex methods may be the best one at this moment.

Definition 4 The local rotational axis is defined as the real eigenvector of the velocity gradient tensor.

The rotational axis is an important element of a vortex and only Liutex/Rortex methods can provide the vortex axis while all other vortex identification methods can only provide a scalar for vortex iso-surfaces and fail to show the vortex vectors and rotational axes.

Definition 5 The vortex core center is defined as a region with local maxima of the relative strength.

Apparently, only the Omega method can give a reasonable answer that vortex core centers are located in the local maxima of Ω. All other methods may miss many vortex cores if the absolute vortex strength is below the threshold. Therefore, when we increase the threshold to find the vortex core centers, only the Omega method can provide the correct vortex core structures while all other methods fail as the threshold remove all weak vortices below the threshold if iso-surfaces are used to describe vortex structures. Multiple thresholds of iso-surfaces only provide chaos but nothing else.

Definition 6 The vortex core size is defined as a place where the relative strength is declined to 95%.

There is only Omega method which can recognize the vortex core size. All other vortex identification methods already lost all weak vortices below the threshold and therefore cannot identify either the vortex core centers or the vortex core sizes of weak vortices.

Definition 7 The boundary of a vortex is defined as a separation line/surface between rotation and non-rotation areas.

Theoretically, $Q>0$, $\Delta>0$ and $\Omega>0.50$ can find the boundary of a vortex, but, in practice, a threshold must be selected to avoid noises. Therefore, the vortex boundary must be set as $Q>Q_{\text{threshold}}$, $\Delta>\Delta_{\text{threshold}}$ and $\Omega>0.51\text{-}0.52$. As addressed above, when the threshold is set to zero, all vortex identification methods demonstrate similar mathematical boundaries and when the threshold is small, all vortex identification methods can still give similar vortex structures due to the continuity of the scalar vortex identification functions. When the threshold is large, only the Omega method can keep the shape of the vortex structure while all other methods wipe all weak vortices out. On the other hand, $Q_{\text{threshold}}$ is strongly case-dependent, must be dramatically changed case by case and cannot give the right vortex boundary when vortices with different strengths co-exist even in the same case. On the other hand, $\Omega>0.51\text{-}0.52$ can be a universal threshold for all cases and all timesteps since only Ω represents the relative vortex strength.

Looking back at the history of vortex identification method development, we can find the first generation of vortex identification method, or vorticity-based methods, fail to answer all above questions since a vortex cannot be represented by the vorticity and the use of the vorticity in general cannot even detect either vortex cores or boundaries. The second generation of vortex identification methods including Q, Δ, λ_2 and λ_{ci} are successful in capturing the approximate vortex boundaries only if the threshold is extremely small but fail to answer all the other critical questions. A vortex has its strength and rotational axis. A vector is required to describe a vortex, not just scalar. All the second generation of vortex identification methods fail to provide a vortex vector. In order to answer questions 4-6, a proper threshold is needed, but no one knows what the proper threshold is. There is no proper threshold for vortex structures if the strong vortices and weak vortices co-exist inside the flow field. The third generation of vortex identification methods, including Omega, Liutex/Rortex, Omega-Liutex, Objective Omega, and Modified Omega, are basically successful in answering all these critical questions. First, all the

Table 1 Capability of major vortex identification methods

Methods	Authors	Vortex boundary with small threshold	Threshold-insensitive	Exact rotation strength (angular speed)	Weak vortex cores	Vortex axis
Vorticity	Helmholtz[19]	No	No	No	No	No
Q criterion	Hunt et al.[4]	Yes	No	No	No	No
Δ criterion	Chong et al.[5]	Yes	No	No	No	No
λ_2 criterion	Jeong et al.[6]	Yes	No	No	No	No
λ_{ci} criterion	Zhou et al.[7]	Yes	No	No	No	No
Omega	Liu et al.[9]	Yes	Yes	No	Yes	No
Liutex/Rortex	Liu et al.[13]	Yes	No	Yes	No	Yes
Omega-Liutex	Dong et al.[15]	Yes	Yes	Yes	Yes	Yes

third-generation methods are successful in capturing the vortex cores. The Omega method is more successful in answering question 2, questions 4-6. The local maximum of Ω could be thought as the core center of vortices. The size of vortex cores could be the location where Ω is declined to 95%. Mathematically, the boundary of a vortex should be located at $\Omega = 0.50$. In practice, the boundary of a vortex could be set $\Omega = 0.51 - 0.52$ and $\Omega = 0.52$ is set by most users. The boundary of vortex could be understood in a sense that there is no more rotation (close streamline or swirling streamline) outside the vortex boundary. In order to answer question 1 to get the angular speed, we must find a rigid rotation part of the fluid motion. Liutex/Rortex is the unique, accurate and best answer while all other vortex identifications like Q, Δ and λ_{ci} are contaminated by shear. In order to capture both strong and weak vortices, the relative strength must be applied. The Omega method which use a ratio of the vorticity squared over the sum of the vorticity squared and the deformation squared is the most successful in capturing both strong and weak vortices simultaneously. However, Ω could be contaminated by shear as well. In contrast, the Omega -Liutex/Rortex method may be the most reasonable one since it is based on a ratio of 2-D vorticity squared over the sum of 2-D vorticity squared and deformation squared in a plane orthogonal to the Liutex/Rortex axis. Question 3 can only be answered by Liutex/ Rortex since all other methods are scalar criteria and fail to provide vectors. Liutex/Rortex is a new mathematical vector definition to describe the rigid rotation of the fluid motion first time in the history of vortex science. Liutex/Rortex has its own scalar (magnitude or angular speed), vector (rotational axis) and tensor (for fluid motion decomposition) forms.

The Objective Omega and objective Liutex/ Rortex methods are under development and will be published very soon. Table 1 shows the capability of the first, second and third generations of vortex identification methods. Apparently, the first generation failed to answer all 6 questions listed above. The second generation can capture strong vortex cores but would fail to capture weak vortex cores if a large threshold is set and some weak vortices are below the threshold. In contrast, Omega and Omega-Liutex can capture both strong and weak vortices if they co-exist. In addition, Liutex/Rortex is a vector which can show both rotational direction and rotational strength and can show Liutex/Rortex vectors and lines, while all other vortex identification methods can only show the iso-surface of vortices as scalar criteria. Scalar quantity may not be good enough to describe vortex structures. In this aspect, Liutex/Rortex is a breakthrough in vortex science.

8. Conclusions

Based on the above descriptions, the following conclusions can be summarized:

(1) The vorticity is defined as the curl of the velocity vector, but a vortex is a natural phenomenon observed as the rotational/swirling motion of fluids. They are different concepts. The vorticity cannot be applied to identify vortices in general.

(2) The currently popular vortex identification methods including Q, Δ, λ_2 and λ_{ci} are based on the Cauchy-Stokes decomposition and/or eigenvalues of the velocity gradient tensor and have several shortcomings: 1) the physical meaning is not very clear, 2) contaminated by shear, 3) very sensitive to threshold change and requires an appropriate threshold which no one knows a priori, 4) cannot capture the strong vortices and weak vortices simultaneously, 5) have only a scalar form for iso-surface visualization without vector and tenser forms. In principle, they cannot find the vortex core structure when they fail to capture all weak vortices below the specified threshold. They keep failed to capture the vortex cores as the threshold is enlarged.

(3) Ω is a measurement of the relative vortex strength and the Omega method has several advantages over other methods: 1) the physical meaning is clear, 2) easy to be implemented, 3)

non-dimensional and normalized from 0 to 1, 4) robust to moderate threshold change, 5) capable to capture both strong and weak vortices simultaneously, 6) correctly detect vortex cores.

(4) Liutex/Rortex is a new mathematical definition for the absolute rigid rotation strength of the fluid motion, which is local, accurate, unique, systematical and Galilean invariant. Liutex/Rortex has its scaler, vector and tensor forms. Liutex/Rortex is an accurate definition for the rigid rotation part of fluid motion and a promising tool to identify vortices. Actually, the magnitude of Liutex/Rortex represents the exact angular speed of the rigid rotation of the fluid motion.

(5) Omega-Liutex/Rortex has advantages of both Ω and Liutex/Rortex.

(6) The vorticity cannot represent the flow rotation but can be decomposed as a rigid rotation part and an antisymmetric shear part, or $\nabla \times \boldsymbol{V} = \boldsymbol{R} + \boldsymbol{S}$.

(7) The velocity gradient tensor should be decomposed as a rotational part and non-rotational part or $\nabla \boldsymbol{V} = \boldsymbol{R} + \boldsymbol{NR}$, based on Liutex/Rortex.

(8) Objectivity is important for turbine machinery and other rotating equipment. The objective Omega and objective Liutex/Rortex method are under development.

(9) A vortex has several critical characters to be specified as: 1) the absolute strength, 2) the relative strength 3) the rotational axis and vortex direction, 4) vortex core center location, 5) vortex core size, 6) vortex boundary.

(10) While most vortex identification methods can identify the vortex boundary when the threshold is close to zero, all of the second generation of the vortex identification methods fail to answer all the above critical questions except for the approximate vortex boundary, but very sensitive to threshold change.

(11) The Omega method as the third generation of the vortex identification methods can well answer all the above critical questions except for questions 1), 3). Liutex/Rortex provides an accurate quantity to represent the absolute vortex rotation strength while most vortex identification methods could be contaminated by shear. Therefore, only Liutex/Rortex can represent the absolute strength. Liutex/Rortex provides the local rotational axis as well and it is the only one showing both the vortex magnitude and rotational axis. Liutex/Rortex vectors and lines will offer a new perspective to analyze vortex structures while no counterpart exists with other vortex identification methods.

(12) It is promising to solve all of the mentioned critical questions by the newly proposed Omega -Liutex method.

(13) Since vortices are the building blocks and driven forces of turbulence, we should have no serious scientific research on the mechanism of turbulence generation and turbulence coherent structure when we do not have an accurate definition of a vortex. We really did not have a definition of a vortex in the past before the third generation of vortex identification was born when we used the first and second generations of vortex identification methods. Therefore, the third generation of vortex identification will open a new era for scientific research of turbulence. A vortex is no more just visualized by some iso-surfaces but has its unique and systematic mathematical definition with six core elements including its local rotational direction and strength, vortex cores, group rotation axes, and vortex boundaries, and these will pave the foundation for scientific research of turbulence.

As the six core elements of the vortex have been mathematically defined by the third generation of vortex identification, it is anticipated that the new vortex dynamics or new Omega-Liutex dynamics will be established, which is extremely important to fluid dynamics, especially critical to fundamental research on turbulence including the turbulence generation, sustenance, structure, modeling and control. It is also anticipated that the new vortex identification methods will be widely applied in variety of scientific research and engineering applications including aerodynamics, hydrodynamics, meteorology, oceanography, metallurgy, astronomy, space science as well as the wind turbine, water turbine, etc..

The software of the third generation of vortex identification methods has been published on line at **https://www.uta.edu/math/cnsm/public_html/cnsm/cnsm.html** for free download with a short agreement for users to sign.

Although the third generation of vortex identification methods including Omega, modified Omega, Liutex/Rortex, Omega-Liutex, objective Omega, objective Liutex have made a breakthrough in vortex identification, there are still many questions remained for researchers to look for more accurate and efficient vortex identification methods.

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